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Operator inequalities of Malamud and Wielandt

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Abstract

We show that the Wielandt operator inequality and the Malamud one are equivalent and discuss some variations of them. From this point of view, we give also a proof of Malamud's multivariable inequality with its variations. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Throughout this note, we discuss operator inequalities for positive operators on a Hilbert space H. For positive invertible operator A with $0 < m \leqslant A \leqslant M$, put

$$m(A) = \left\|A^{-1}\right\|^{-1} = \min\{t \mid t \in \sigma(A)\} \geqslant m,$$

$$M(A) = \|A\| = \max\{t \mid t \in \sigma(A)\} \leqslant M.$$

Then we have

$$\frac{M-m}{M+m} \geqslant \frac{M(A)-m(A)}{M(A)+m(A)}.$$

Here we call the constant

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$$\left(\frac{M(A)-m(A)}{M(A)+m(A)}\right)^2$$

Wielandt's constant for A which is denoted by $W_m(A)$. So, as in [1], the Wielandt inequality is represented as follows:

Wielandt Theorem. *If A is a positive invertible operator, then*

$$|\langle Ay, x\rangle|^2 \leqslant W_m(A)\langle Ax, x\rangle\langle Ay, y\rangle$$

for every orthogonal pair x and y.

As in [2, Theorem 7.4.34], this inequality is an improvement of the Cauchy–Schwarz one. Also, as in [1], it is an extension of the Kantorovich inequality. Thus the Wielandt inequality is an important and fundamental operator one. Note that this inequality is best possible, which we will also show later.

On the other hand, Malamud showed in [3, Theorem 1] an operator inequality related to Wielandt's constant $W_m(A)$:

Malamud Theorem. *If A is a positive invertible operator, then the following properties for positive numbers* α *are equivalent*:

(M) $A + (\alpha - 1)PAP \geqslant 0$ *for all projections P*;

(A) $\alpha \geqslant W_m(A)$.

Although his original theorem is the case that (M) and (A) are strict inequalities '$>$' corresponding to both positivity and invertibility of operator matrices, we treat them as inequalities '$\geqslant$' corresponding to positivity alone throughout this note for the sake of convenience.

In [3], he also showed a multivariable version of this theorem, which is not an extension in the exact sense. We think that the difficulty in obtaining a strict multivariable extension lies in the difference between 2×2 matrices and 3×3 ones (see Lemma 4).

In this note, we show that the Malamud theorem implies the Wielandt one so that they are essentially equivalent. To fill the gap between them, we consider equivalent matrix representations to (W) and (A). Considering such matrix representations, we give a simple proof of Malamud's multivariable inequality, which might show the above difficulty.

2. Equivalence theorem

First we cite the following lemma. Although it may be known, we prove it for completeness.

Lemma 1. *An operator A on H is positive if and only if*

$$A[x_1, \ldots, x_n] \equiv \begin{pmatrix} \langle Ax_1, x_1\rangle & \langle Ax_2, x_1\rangle & \cdots & \langle Ax_n, x_1\rangle \\ \langle Ax_1, x_2\rangle & \langle Ax_2, x_2\rangle & & \langle Ax_n, x_2\rangle \\ \vdots & & \ddots & \vdots \\ \langle Ax_1, x_n\rangle & \langle Ax_2, x_n\rangle & \cdots & \langle Ax_n, x_n\rangle \end{pmatrix} \geqslant 0$$

for all natural numbers n and all mutually orthogonal vectors $x_k \in H$ $(k = 1, \ldots, n)$, *or equivalently,*

$$\begin{pmatrix} P_1AP_1 & P_1AP_2 & \cdots & P_1AP_n \\ P_2AP_1 & P_2AP_2 & & P_2AP_n \\ \vdots & & \ddots & \vdots \\ P_nAP_1 & P_nAP_2 & \cdots & P_nAP_n \end{pmatrix} \geqslant 0$$

for all n and all mutually orthogonal projections P_k.

Proof. For a positive operator A, the determinant of the matrix $A[x_1, \ldots, x_n]$ is nothing but the Gramian for $\{A^{1/2}x_1, \ldots, A^{1/2}x_n\}$, so that it is positive. Conversely if $A[x_1, \ldots, x_n]$ is positive, then the inequality

$$\begin{aligned} &\langle A(x_1 \oplus \cdots \oplus x_n), x_1 \oplus \cdots \oplus x_n\rangle \\ &\quad = \left\langle \begin{pmatrix} \langle Ax_1, x_1\rangle & \langle Ax_2, x_1\rangle & \cdots & \langle Ax_n, x_1\rangle \\ \langle Ax_1, x_2\rangle & \langle Ax_2, x_2\rangle & & \langle Ax_n, x_2\rangle \\ \vdots & & \ddots & \vdots \\ \langle Ax_1, x_n\rangle & \langle Ax_2, x_n\rangle & \cdots & \langle Ax_n, x_n\rangle \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle \geqslant 0 \end{aligned}$$

implies $A \geqslant 0$. $\square$

Remark 1. Lemma 1 is valid also if the words 'for all natural numbers n' is exchanged for 'for some natural number n'. In particular, considering the case $n = 2$, A is positive if and only if

$$\langle Ax, x\rangle\langle Ay, y\rangle - |\langle Ax, y\rangle|^2 = \det A[x, y] \geqslant 0$$

for all orthogonal vectors x and y.

We now show the equivalence theorem including the Malamud and the Wielandt ones. For completeness, we show it without the Wielandt inequality.

Theorem 2. *If A is a positive invertible operator on H, then the following conditions are equivalent*:

(A) $\alpha \geqslant W_m(A)$;

(M) $A + (\alpha - 1)PAP \geqslant 0$ *for all projections P*;

(M0) $\begin{pmatrix} \alpha PAP & PAP^\perp \\ P^\perp AP & P^\perp AP^\perp \end{pmatrix} \geqslant 0$ *for all projections P*;

(M1) $\begin{pmatrix} \alpha PAP & PAQ \\ QAP & QAQ \end{pmatrix} \geqslant 0$ *for all mutually orthogonal projections P and Q*;

(M2) $\begin{pmatrix} \alpha\langle Ax, x\rangle & \langle Ay, x\rangle \\ \langle Ax, y\rangle & \langle Ay, y\rangle \end{pmatrix} \geqslant 0$ *for all orthogonal pair x and y*;

(W) $|(Ay, x)|^2 \leqslant \alpha(Ax, x)(Ay, y)$ *for every orthogonal pair x and y.*

Proof. Lemma 1 with Remark 1 shows the equivalence of (M1), (M2) and (W). Identifying an operator A on H as

$$\begin{pmatrix} PAP & PAP^{\perp} \\ P^{\perp}AP & P^{\perp}AP^{\perp} \end{pmatrix}$$

on $PH \oplus P^{\perp}H$, we have $A + (\alpha - 1)PAP$ corresponds with the operator matrix in (M0). Thus (M) and (M0) are equivalent. If P and Q are mutually orthogonal, then $Q \leqslant P^{\perp}$ and $QP^{\perp} = Q$, and hence the equation

$$\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} \alpha PAP & PAP^{\perp} \\ P^{\perp}AP & P^{\perp}AP^{\perp} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} \alpha PAP & PAQ \\ QAP & QAQ \end{pmatrix}$$

assures that (M0) implies (M1), which shows the equivalence.

Now we only have to show that the Wielandt inequality in the above is best possible: since $m = m(A)$ and $M = M(A)$ belong to the approximate point spectrum of A, then there exist sequences $\{x_n\}$ and $\{y_n\}$ of mutually orthogonal unit vectors which satisfy $\|(A - M)x_n\| \to 0$ and $\|(A - m)y_n\| \to 0$ as $n \to \infty$. Thereby

$$|\langle Ax_n, y_n\rangle| = |\langle (A - m)x_n, y_n\rangle| \leqslant \|(A - m)x_n\| \to 0.$$

Put $u_n = x_n + y_n$ and $v_n = x_n - y_n$. Then we have $\|u_n\| = \|v_n\| = \sqrt{2}$ and $\langle u_n, v_n\rangle = 0$. Therefore it follows that

$$\begin{aligned} &|(Au_n, u_n) - (M + m)| \\ &\quad \leqslant \sqrt{2}\|(A - M)x_n\| + \sqrt{2}\|(A - m)y_n\| + |\langle Ax_n y_n\rangle + \langle Ay_n, x_n\rangle| \to 0 \end{aligned}$$

as $n \to 0$. Similarly, we have $(Av_n, v_n) \to M + m$ and $(Av_n, u_n) \to M - m$. By the assumption of (W), we have $|(Av_n, u_n)|^2 \leqslant \alpha(Av_n, v_n)(Au_n, u_n)$ for all n, and hence we have $((M - m)/2)^2 \leqslant \alpha((M + m)/2)^2$ as the limit. $\square$

Remark 2. As one can observe from the above proof, Theorem 2 is valid if all the projections are assumed to be of rank 1. Incidentally, one can show that condition (M) is exchanged to

(Mc) $A + (\alpha - 1)W^*AW \geqslant 0$ for all contractions W.

Since

$$\begin{pmatrix} \alpha\beta A & C \\ C^* & B \end{pmatrix} \geqslant 0 \quad \text{if and only if} \quad \begin{pmatrix} \alpha A & C \\ C^* & \beta B \end{pmatrix} \geqslant 0,$$

we have the following 2-variable version of Theorem 2:

Corollary 3. *If A is a positive invertible operator on H, then the following properties are equivalent for positive numbers α and β:*

(A′) $\alpha\beta \geqslant W_m(A)$;

(M′) $A + (\alpha - 1)PAP + (\beta - 1)P^\perp AP^\perp \geqslant 0$ *for all projections P*;

(M0′) $\begin{pmatrix} \alpha PAP & PAP^\perp \\ P^\perp AP & \beta P^\perp AP^\perp \end{pmatrix} \geqslant 0$ *for all projections P*;

(M1′) $\begin{pmatrix} \alpha PAP & PAQ \\ QAP & \beta QAQ \end{pmatrix} \geqslant 0$ *for all mutually orthogonal projections P and Q*;

(M2′) $\begin{pmatrix} \alpha\langle Ax, x\rangle & \langle Ay, x\rangle \\ \langle Ax, y\rangle & \beta\langle Ay, y\rangle \end{pmatrix} \geqslant 0$ *for all orthogonal pair x and y*;

(W′) $|(Ay, x)|^2 \leqslant \alpha\beta(Ax, x)(Ay, y)$ *for every orthogonal pair x, y.*

Remark 3. In the above corollary, $\alpha \leqslant 1$ and $\beta \leqslant 1$ are not assumed. This is a slight difference between Malamud's approach and ours. In fact, under the assumptions $\alpha \leqslant 1$ and $\beta \leqslant 1$, Malamud showed that the following condition is also equivalent in [3, Theorem 3]:

(M″) $A + (\alpha - 1)PAP + (\beta - 1)QAQ \geqslant 0$ for all mutually orthogonal projections P and Q,

or equivalently, for $R = 1 - (P + Q)$,

$$\begin{pmatrix} RAR & RAP & RAQ \\ PAR & \alpha PAP & PAQ \\ QAR & QAP & \beta QAQ \end{pmatrix} \geqslant 0,$$

which is the bridge between this section and the following one.

3. Multivariable Malamud inequality

It seems difficult for us to extend the Malamud inequality to its multivariable version. As a matter of fact, consider the following matrices for $0 < m < 1 < M$:

$$B_k = \begin{pmatrix} M+m & M-m & \cdots & M-m \\ M-m & M+m & & M-m \\ \vdots & & \ddots & \vdots \\ M-m & \cdots & M-m & M+m \end{pmatrix}$$

and

$$C_\ell = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

where B_k (resp., C_ℓ) is a $k \times k$ (resp., $\ell \times \ell$) matrix. Then the direct sum $A = B_k + C_\ell$ satisfies $m \leqslant A \leqslant M$ and the critical values α_i for a *multivariable α-condition*

$$\begin{pmatrix} \alpha_1(M+m) & M-m & \cdots & M-m & & & \\ M-m & \alpha_2(M+m) & & M-m & & \mathbf{0} & \\ \vdots & & \ddots & \vdots & & & \\ M-m & \cdots & M-m & \alpha_k(M+m) & & & \\ & & & & \alpha_{k+1} & \cdots & 1 \\ & \mathbf{0} & & & \vdots & \ddots & \vdots \\ & & & & 1 & \cdots & \alpha_{k+\ell} \end{pmatrix} \geqslant 0$$

are $\alpha_j = (M-m)/(M+m)$ for $1 \leqslant j \leqslant k$ and $\alpha_j = 1$ otherwise. So we have

$$\prod_{j=1}^{k+\ell} \alpha_j = \left(\frac{M-m}{M+m}\right)^k.$$

Considering this example, the relation between the multivariable α-condition in the above sense and Wielandt's constant is ambiguous when we consider only the size $n = k + \ell$ of A and the bounds m and M of A. Nevertheless, Malamud showed a very ingenious multivariable formula. But the original proof is too complicated for us to consider whether we can improve his theorem or not, so we reconstruct his proof. As in the preceding section, the following theorem is equivalent to his inequality in [3, Theorem 2]:

Malamud Multivariable Theorem. *Let A be a positive invertible operator with $0 < m \leqslant A \leqslant M$ for $m < M$. If*

$$\alpha_1 \cdots \alpha_n \geqslant \left(\frac{M-m}{M+m}\right)^2 \quad \textit{for } \alpha_k \in (0,1],$$

then

$$\begin{pmatrix} \alpha_1\langle Ax_1, x_1\rangle & \langle Ax_2, x_1\rangle & \cdots & \langle Ax_n, x_1\rangle \\ \langle Ax_1, x_2\rangle & \alpha_2\langle Ax_2, x_2\rangle & & \langle Ax_n, x_2\rangle \\ \vdots & & \ddots & \vdots \\ \langle Ax_1, x_n\rangle & \langle Ax_2, x_n\rangle & \cdots & \alpha_n\langle Ax_n, x_n\rangle \end{pmatrix} \geqslant 0$$

for all mutually orthogonal vectors x_j, or equivalently,

$$\begin{pmatrix} \alpha_1 P_1AP_1 & P_1AP_2 & \cdots & P_1AP_n \\ P_2AP_1 & \alpha_2 P_2AP_2 & & P_2AP_n \\ \vdots & & \ddots & \vdots \\ P_nAP_1 & P_nAP_2 & \cdots & \alpha_n P_nAP_n \end{pmatrix} \geqslant 0$$

for all mutually orthogonal projections P_k.

To clear his approach, we cite two lemmas. The first lemma shows a gap between 2×2 matrices and 3×3 ones. Note that the assumptions $t \leqslant 1$ and $s \leqslant 1$ are necessary.

Lemma 4. *For positive semidefinite* 3×3 *matrix* $A = (a_{ij})$ *and numbers* $0 < t, s \leqslant 1$, *if*

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & sta_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \textit{and} \quad A_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & sta_{33} \end{pmatrix}$$

are positive semidefinite, so is

$$A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & sa_{22} & a_{23} \\ a_{31} & a_{32} & ta_{33} \end{pmatrix}.$$

Proof. We may assume that $a_{22}|a_{13}|^2 \leqslant a_{33}|a_{12}|^2$. Then we have

$$\begin{aligned} \det\ A_3 - \det\ A_1 &= sta_{22}|a_{13}|^2 + a_{33}|a_{12}|^2 - \left(sa_{22}|a_{13}|^2 + ta_{33}|a_{12}|^2\right) \\ &= (1-t)\left(a_{33}|a_{12}|^2 - sa_{22}|a_{13}|^2\right) \\ &\geqslant (1-t)\left(a_{33}|a_{12}|^2 - a_{22}|a_{13}|^2\right) \geqslant 0. \end{aligned}$$

By $\det\ A_1 \geqslant 0$, we have $\det\ A_3 \geqslant 0$ and hence $A_3 \geqslant 0$. $\square$

The following lemma is the heart of Malamud's proof:

Lemma 5. *Let* $0 < m \leqslant A \leqslant M$ *for* $m < M$. *If*

$$\alpha_2 \cdots \alpha_n \geqslant \left(\frac{M-m}{M+m}\right)^2 \quad \textit{for } \alpha_k \in (0, 1],$$

then

$$\begin{pmatrix} P_1AP_1 & P_1AP_2 & \cdots & P_1AP_n \\ P_2AP_1 & \alpha_2 P_2AP_2 & & P_2AP_n \\ \vdots & & \ddots & \vdots \\ P_nAP_1 & P_nAP_2 & \cdots & \alpha_n P_nAP_n \end{pmatrix} \geqslant 0$$

for all mutually orthogonal projections P_k.

Proof. Since it holds for the case $n = 2$, we have only to show the case $n = k + 1$ under the assumption of case $n = k$. By the assumption of this induction, we have

$$A_1 \equiv \begin{pmatrix} P_1AP_1 & P_1AP_2 & & \cdots & P_1AP_{k+1} \\ P_2AP_1 & \alpha_2P_2AP_2 & & & P_2AP_{k+1} \\ \vdots & & \ddots & & \vdots \\ P_kAP_1 & P_kAP_2 & \cdots & \alpha_k\alpha_{k+1}P_kAP_k & P_kAP_{k+1} \\ P_{k+1}AP_1 & P_{k+1}AP_2 & & \cdots & P_{k+1}AP_{k+1} \end{pmatrix} \geqslant 0$$

and

$$A_2 \equiv \begin{pmatrix} P_1AP_1 & P_1AP_2 & & \cdots & P_1AP_{k+1} \\ P_2AP_1 & \alpha_2P_2AP_2 & & & P_2AP_{k+1} \\ \vdots & & \ddots & & \vdots \\ P_kAP_1 & P_kAP_2 & \cdots & P_kAP_k & P_kAP_{k+1} \\ P_{k+1}AP_1 & P_{k+1}AP_2 & & \cdots & \alpha_k\alpha_{k+1}P_{k+1}AP_{k+1} \end{pmatrix} \geqslant 0$$

since they are reduced to the case $n = k$ by putting $P_1' = P_1 + P_{k+1}$ and $P_1'' = P_1 + P_k$, respectively:

$$A_1 = \begin{pmatrix} P_1'AP_1' & P_1'AP_2 & \cdots & P_1'AP_k \\ P_2AP_1' & \alpha_2P_2AP_2 & & P_2AP_k \\ \vdots & & \ddots & \vdots \\ P_kAP_1 & P_kAP_2 & \cdots & \alpha_k\alpha_{k+1}P_kAP_k \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} P_1''AP_1'' & P_1''AP_2 & \cdots & P_1''AP_{k-1} & P_1''AP_{k+1} \\ P_2AP_1 & \alpha_2P_2AP_2 & & P_2AP_{k-1} & P_2AP_{k+1} \\ \vdots & & \ddots & & \vdots \\ P_kAP_1 & P_kAP_2 & \cdots & P_{k-1}AP_{k-1} & P_{k-1}AP_{k+1} \\ P_{k+1}AP_1 & P_{k+1}AP_2 & \cdots & P_{k+1}AP_{k-1} & \alpha_k\alpha_{k+1}P_{k+1}AP_{k+1} \end{pmatrix}.$$

Here we want to show that

$$A_0 \equiv \begin{pmatrix} P_1AP_1 & P_1AP_2 & & \cdots & P_1AP_{k+1} \\ P_2AP_1 & \alpha_2P_2AP_2 & & & P_2AP_{k+1} \\ \vdots & & \ddots & & \vdots \\ P_kAP_1 & P_kAP_2 & \cdots & \alpha_kP_kAP_k & P_kAP_{k+1} \\ P_{k+1}AP_1 & P_{k+1}AP_2 & & \cdots & \alpha_{k+1}P_{k+1}AP_{k+1} \end{pmatrix} \geqslant 0.$$

Putting $Q_1 = P_1 + \cdots + P_{k-1}$, $Q_2 = P_k$ and $Q_3 = P_{k+1}$, we observe that

$$A_1 = \begin{pmatrix} Q_1A_0Q_1 & Q_1A_0Q_2 & Q_1A_0Q_3 \\ Q_2A_0Q_2 & \alpha_k\alpha_{k+1}Q_2A_0Q_2 & Q_2A_0Q_3 \\ Q_3A_0Q_1 & Q_3A_0Q_2 & Q_3A_0Q_3 \end{pmatrix} \geqslant 0$$

and

$$A_2 = \begin{pmatrix} Q_1A_0Q_1 & Q_1A_0Q_2 & Q_1A_0Q_3 \\ Q_2A_0Q_1 & Q_2A_0Q_2 & Q_2A_0Q_3 \\ Q_3A_0Q_1 & Q_3A_0Q_2 & \alpha_k\alpha_{k+1}Q_3A_0Q_3 \end{pmatrix} \geqslant 0.$$

Noting that

$$\alpha_k\alpha_{k+1} \geqslant \alpha_2 \cdots \alpha_k \geqslant \left(\frac{M-m}{M+m}\right)^2,$$

we have

$$A = \begin{pmatrix} Q_1A_0Q_1 & Q_1A_0Q_2 & Q_1A_0Q_3 \\ Q_2A_0Q_1 & \alpha_k Q_2A_0Q_2 & Q_2A_0Q_3 \\ Q_3A_0Q_1 & Q_3A_0Q_2 & \alpha_{k+1}Q_3A_0Q_3 \end{pmatrix} \geqslant 0$$

since what we claim here is reduced to Lemma 4 by Lemma 1. $\square$

Although this lemma is the special case $\alpha_1 = 1$ of the above theorem, it also shows the theorem as positivity of submatrices in it.

Remark 4. Alhough Malamud's multivariable theorem is a nice one, the condition $\alpha_1 \cdots \alpha_n \geqslant W_m(A)$ for $\alpha_k \in (0, 1]$ is a merely sufficient one for multivariable Malamud inequality since the inequality may hold for $\alpha_k > 1$ for some *k*. In fact, consider a matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

For $\alpha_1 = 1/4$ and $\alpha_2 = \alpha_3 = 4/3 > 1$, we have $1 \leqslant m(A) \leqslant A \leqslant M(A) \leqslant 5$ and

$$\begin{pmatrix} 4\alpha_1 & 1 & 1 \\ 1 & 2\alpha_2 & 1 \\ 1 & 1 & 2\alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 8/3 & 1 \\ 1 & 1 & 8/3 \end{pmatrix} \geqslant 0$$

and hence

$$W_m(A) \leqslant \left(\frac{5-1}{5+1}\right)^2 = \alpha_1\alpha_2\alpha_3.$$

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